HEAT EXCHANGE OF ELLIPTICAL CYLINDERS SURROUNDED BY A GAS FLOW AT LOW PECLET NUMBERS

I. M. Skachkov, E. R. Shchukin
and Yu. I. Yalamov

UDC 533.72

Heat exchange of a circular cylinder with surrounding gas flow was studied in detail in [1-3]. The temperature and Nusselt number were obtained to an accuracy of Pe^2 for low finite Peclet and Reynolds numbers in [2, 3]. Heat exchange of noncircular cylinders has not been considered previously. The present study will examine heat exchange of an elliptical cylinder with a gas flow perpendicular to the cylinder generatrix.

We will consider the temperature distribution T_i along the cylinder surface to be homogeneous (T_i = const). We consider the case Pe $\ll$ 1 and low relative temperature differentials T in the cylinder-gas flow system ($|T_i - T|/T \ll 1$).

To find the temperature distribution T in the gas flow we will solve the Ozeen equation, which, as was shown in [2, 3], gives a valid zeroth approximation for T:

$$\mathbf{u}\,\mathrm{grad}\,T = \chi\,\mathrm{div}\,\mathrm{grad}\,T \tag{1}$$

with boundary conditions

$$T = T_i \ \text{on the cylinder surface}, \qquad T = T_\infty \ \text{at infinity}, \tag{2}$$

where χ is the thermal diffusivity. It is easiest to perform the solution in an elliptical coordinate system, related to the Cartesian coordinates by the following expressions:

$$x = c\,\mathrm{ch}\,\xi\cos\eta, \quad y = c\,\mathrm{sh}\,\xi\sin\eta,$$

where c is the ellipse focus distance. Transforming from the dependent variable T to the dimensionless variable $t = (T - T_\infty)/(T_i - T_\infty)$ and performing the substitution $t = \exp[(k/a)(x\cos\eta_0 + y\sin\eta_0)]v$, from Eqs. (1) and (2) we arrive at the simpler system:

$$\frac{\partial^2 v}{\partial\xi^2} + \frac{\partial^2 v}{\partial\eta^2} + (kc/a)^2(\cos^2\eta - \mathrm{ch}^2\xi)\,v = 0; \tag{3}$$

$$v(\xi_0, \eta) = \exp[-(k/a)(a\cos\eta_0\cos\eta - b\sin\eta_0\sin\eta)], \quad v \to 0 \ \text{for} \ \xi \to \infty, \tag{4}$$

where a and b are the major and minor semiaxes of the ellipse; $k = \frac{au}{2\chi} = \frac{1}{4}\mathrm{Pe}$; η_0 is the angle formed by the velocity $\mathbf{u}$, with the major semiaxis a. We obtain as a solution for Eq. (3) an expression for the function $v = v(\xi, \eta)$

$$\begin{aligned} v &= \sum_{n=0}^{\infty}(\gamma_n \mathrm{Cek}_n(\xi)\,\mathrm{ce}_n(\eta) + \omega_n \mathrm{Sek}_n(\xi)\,\mathrm{se}_n(\eta)), \\ \mathrm{ce}_{2n+i}(\eta) &= \sum_{r=0}^{\infty} A_{2r+i}^{2n+i}\cos(2r+i)\eta, \\ \mathrm{se}_{2n+i}(\eta) &= \sum_{r=0}^{\infty} B_{2r+i}^{2n+i}\sin(2r+i)\eta, \\ \mathrm{Cek}_{2n+i}(\xi) &= \sum_{r=0}^{\infty} A_{2r+i}^{2n+i} K_{2r+i}\left(k\frac{c}{a}\mathrm{ch}\,\xi\right), \\ \mathrm{Sek}_{2n+i}(\xi) &= \mathrm{th}\,\xi\sum_{r=0}^{\infty}(2r+i)\,B_{2r+i}^{2n+i}\left(k\frac{c}{a}\mathrm{ch}\,\xi\right), \end{aligned} \tag{5}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 31-34, January-February, 1982. Original article submitted December 2, 1980.

 0021-8944/82/2301-0026$07.50

$$A_r^r = B_r^r = 1, \quad A_{n+2r}^n = B_{n+2r}^n = \frac{k^{2r} n!}{r!\,(n+r)!\,4^{2r}},$$
$$A_{n-2r}^n = B_{n-2r}^n \frac{(-1)^r (n-1-r)!\,k^{2r}}{r!\,(n-1)!\,4^{2r}}, \quad i = 0, 1,$$

where K_n is a modified Bessel function of the second sort [4]; γ_n and ω_n are arbitrary constants which can be found from the boundary condition on the cylinder surface. Substituting Eq. (5) in Eq. (4), expanding the exponential in a Fourier series, and separating even and odd terms, we obtain four independent infinite algebraic systems of equations for determination of γ_{2n}, γ_{2n+1}, ω_{2n} and ω_{2n+1}:

$$\sum_{n=0}^{\infty} \gamma_{2n} \operatorname{Cek}_{2n} A_{2r}^{2n} = (2 - \delta_{0r}) I_{2r}(z) \cos 2r\varphi; \tag{6}$$

$$\sum_{n=0}^{\infty} \gamma_{2n+1} \operatorname{Cek}_{2n+1} A_{2r+1}^{2n+1} = -2 I_{2r+1}(z) \cos (2r+1)\varphi; \tag{7}$$

$$\sum_{n=0}^{\infty} \omega_{2n+1} \operatorname{Sek}_{2n+1} B_{2r+1}^{2n+1} = -2 I_{2r+1}(z) \sin (2r+1)\varphi; \tag{8}$$

$$\sum_{n=0}^{\infty} \omega_{2n+2} \operatorname{Sek}_{2n+2} B_{2r+2}^{2n+2} = 2 I_{2r+2}(z) \sin (2r+2)\varphi, \tag{9}$$

$$r = 0, 1, 2, \ldots, \quad \delta_{0r} = \begin{pmatrix} 1, & r = 0 \\ 0, & r \neq 0 \end{pmatrix},$$

where

$$\operatorname{Cek}_n = \operatorname{Cek}_n(\xi_0); \quad \operatorname{Sek}_n = \operatorname{Sek}_n(\xi_0); \quad z = (k/a)\sqrt{a^2 \cos^2 \eta_0 + b^2 \sin^2 \eta_0};$$
$$\varphi = \arccos \left(a \cos \eta_0 / \sqrt{a^2 \cos^2 \eta_0 + b^2 \sin^2 \eta_0}\right).$$

Analysis of the behavior of the Matier functions introduced in Eq. (5) at low k values makes possible determination of the order with respect to this parameter of the coefficients for the unknown γ_n and ω_n [4]:

$$\operatorname{Cek}_0 A_0^0 \sim \ln k, \quad \operatorname{Cek}_{r+2n} A_r^{r+2n} \sim \operatorname{Sek}_{r+2n} B_r^{r+2n} \sim k^{-r},$$
$$\operatorname{Cek}_{r-2n} A_r^{r-2n} \sim \operatorname{Sek}_{r-2n} B_r^{r-2n} \sim k^{-r+4n}. \tag{10}$$

We use Cramer's rule to solve Eqs. (6)–(9). It follows from evaluation of Eq. (10) that in the determinants composed of the coefficients of the unknowns, the product of the diagonal elements is much larger than the remaining terms. The determinants obtained by replacing the first columns by a column of free terms have the same property. Calculation of the subsequent unknowns γ_n and ω_n becomes more complicated with increase in n, although for an estimate it is sufficient as before to compare the product of the diagonal elements of corresponding determinants. As a result, the following expressions are obtained:

$$\gamma_0 = [\ln (4a/k\gamma(a+b))]^{-1} + O(k^2),$$
$$|\gamma_n| = O(k^{2n}), \quad |\omega_n| = O(k^{2n}) \quad (n \geqslant 1), \tag{11}$$

where $\ln \gamma (=0,5772\ldots)$ is Euler's constant. Neglecting in Eq. (5) the terms proportional to k, we obtain for $k \ll 1$

$$v \simeq \gamma_0 \operatorname{Cek}_0(\xi) \operatorname{ce}_0(\eta),$$
$$t \simeq \gamma_0 \exp \left[(k/a)(c \operatorname{ch} \xi \cos \eta_0 \cos \eta + c \operatorname{sh} \xi \sin \eta_0 \sin \eta)\right] \operatorname{Cek}_0(\xi) \operatorname{ce}_0(\eta).$$

The heat flux removed from (delivered to) a unit cylinder length is found from the expression

$$Q_T = -\varkappa \oint (\mathbf{n} \operatorname{grad} T - T\mathbf{u}\mathbf{n}/\chi)\, ds, \tag{12}$$

where $\varkappa$ is the thermal conductivity coefficient; **n** is the external normal to the curve along which integration is performed. The simplest form of Eq. (12) occurs as $\xi \to \infty$:

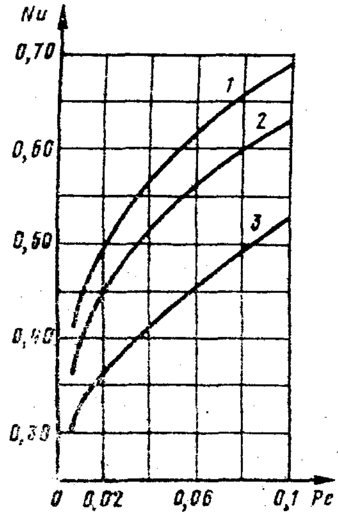

Fig. 1

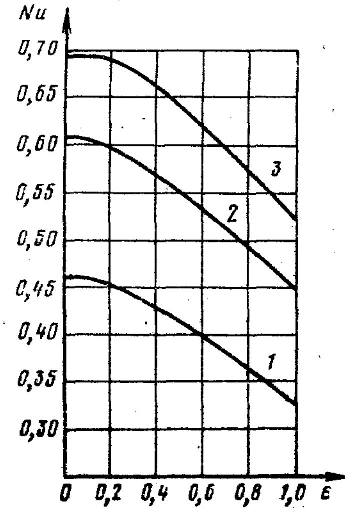

Fig. 2

$$Q_T = \pi \varkappa (T_i - T_\infty) \int_0^{2\pi} \exp [(kc/2a) \exp (\xi) \cos (\eta - \eta_0)] \left[(kc/2a) \exp (\xi) \cos (\eta - \eta_0)\, v - \frac{\partial v}{\partial \xi} \right] d\eta.$$

Integrating, we obtain

$$Q_T = 2\pi \varkappa (T_i - T_\infty) \sum_{n=0}^{\infty} \left[\gamma_n \sum_{r=0}^{\infty} A_r^n \cos n\eta_0 + \omega_{n+1} \sum_{r=0}^{\infty} B_{r+1}^{n+1} \sin (n+1) \eta_0 \right]. \tag{13}$$

In the case of small Peclet numbers Eq. (13) transforms to

$$Q_T = 2\pi\varkappa(T_i - T_\infty)\gamma_0 = \mathrm{Nu}\, \varkappa(T_i - T_\infty) l/2a, \tag{14}$$

where Nu is the Nusselt number. Substituting Eq. (11) in Eq. (14) and transforming from k to the Peclet number, we obtain for the Nusselt number

$$\mathrm{Nu} = (4\pi a/l) [\ln (16a/\mathrm{Pe}\, \gamma(a+b))]^{-1}, \tag{15}$$

where l is the length of the ellipse periphery. As $b/a \to 0$ (case of a lamina) Eq. (15) tends to the limit $\lim_{b/a \to 0} \mathrm{Nu} = \pi (\ln (16/\mathrm{Pe}\, \gamma))^{-1}$ in the limiting case of a circular cylinder (a = b) Eq. (15) transforms to the expression presented in [1]: $\mathrm{Nu} = 2(ln\ (8/\mathrm{Pe}\ \gamma))^{-1}$. It follows from Eq. (15) that in the approximation taken at Pe $\ll 1$ the heat flux removed from the cylinder surface is independent of cylinder orientation and is determined only by the value of Pe and the semiaxis ration b/a.

Curves of Nu as a function of Pe and $\varepsilon = b/a$ are shown in Figs. 1 and 2. In Fig. 1 curves 1-3 are constructed for ε = 0.1, 0.5, and 1, respectively. In Fig. 2 curves 1-3 are constructed for Pe = 0.01, 0.05, and 0.1, respectively.

LITERATURE CITED

1. H. C. Levey, "heat transfer in slip flow at low Reynolds number," J. Fluid Mech., 6, 385 (1959).
2. D. R. Kassoy, "Heat transfer from circular cylinders at low Reynolds number," Phys. Fluids, 10, 939 (1967).
3. C. A. Hieber and B. Gebhart, "Low Reynolds number heat transfer from a circular cylinder," J. Fluid Mech., 32, 21 (1968).
4. Handbook of Special Functions [in Russian], Nauka, Moscow (1979).